ABSTRACT

Title of Dissertation: LIMIT THEOREMS
AND THE KONTSEVICH-ZORICH COCYCLE:

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This thesis concerns the study of Teichmüller dynamics, and in particular the Kontsevich-Zorich cocycle, which is a central object in that theory. In particular, we study and prove limit theorems for this cocycle. In Chapter 2, we present a mechanism for producing oscillations along the lift of the Teichmüller geodesic flow to the (real) Hodge bundle, as the basepoint surface is deformed by a unipotent element of $\mathrm{SL}_2(\mathbb{R})$. We apply our methods to all connected components of strata which exhibit a varying phenomenon for the sum of non-negative Lyapunov exponents. In genus 4, by work of Chen-Möller, eight connected components of strata exhibit varying Lyapunov exponents, and so we apply our methods to those connected components that are shown to be varying by their work. In Chapter 3, which can be read independently of Chapter 2, we show that a central limit theorem holds for the top exterior power of the Kontsevich-Zorich cocycle. In particular, we show that a central limit theorem holds for the the lift of the (leafwise) hyperbolic Brownian motion to the Hodge bundle, and then show that a (possibly degenerate) central limit theorem holds for the the lift of the Teichmüller geodesic flow to the same bundle. We show that the variance of the limiting distribution for the random cocycle is positive if the second top Lyapunov exponent of the cocycle is positive.

LIMIT THEOREMS
AND THE KONTSEVICH-ZORICH COCYCLE

by

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Chapter 1: Introduction

To set the stage for the thesis, suppose that one has a particle moving at unit speed in a polygon. As the particle hits an edge, it changes direction as governed by the law of reflection. The particle ceases to move if it reaches a vertex of a polygon. We call the path that is traversed by this particle a billiard trajectory. Many questions of physical interest can be addressed by understanding the motion of this particle, ranging from illumination problems [1] to studying diffusion rates on wind-tree models [2], among many other examples.

It turns out that if the interior angles of these polygons are rational multiples of π, polygonal billiards give rise to translation surfaces, i.e. flat surfaces with finitely many singularities and trivial linear holonomy. This is accomplished via the so-called unfolding construction of Katok-Zemlyakov [3], which reduces the study of billiard trajectories on rational polygons to straight-line trajectories on translation surfaces.

Suppose then that one is interested in studying the long-term behavior of straight-line trajectories on a translation surface. One can attempt to replace long trajectories on a translation surface on the one hand, with bounded trajectories in a family of translation surfaces on the other hand. This is accomplished by renormalization, with the aim of transferring understanding of the renormalized surfaces to that of the original surface.

This renormalizing dynamical system in our setting is called the Teichmüller geodesic flow, and is a central object in this thesis.

In fact, the thesis concerns the Kontsevich-Zorich cocycle, which is intimately related to the tangent cocycle of the Teichmüller geodesic flow. The associated Lyapunov exponents of this cocycle play a key role in questions related to the deviation of ergodic averages [4] and weak mixing for the straight-line flow on translation surfaces [5]. These exponents measure the exponential growth of tangent vectors as they undergo parallel transport along trajectories of the Teichmüller geodesic flow.

This thesis is composed of two chapters that can be read independently. The topics that are addressed in this thesis are as follows:

- In Chapter 2, we show that the Kontsevich-Zorich cocycle, for a fixed surface, and for almost all directions, cannot be normalized by a function that is independent of the direction, in eight connected components of strata in genus 4, those that exhibit a varying phenomenon as shown in work of Chen-Möller [6].

- In Chapter 3, we show that a central limit theorem holds for the top exterior power of the Kontsevich-Zorich cocycle.

In this introductory chapter, we give an overview of the results and raise questions that arise from some of the answers that we provide, as well as other questions that we hope to address in a future work.

1.1 Results and Questions

In order to state the results presented in this thesis, we will fix some notation and be more precise about the objects described in the introduction. Let S be a Riemann surface of genus $g \geq 2$, and ω a holomorphic 1-form on S. The pair (S, ω) is said to be a translation surface, since ω gives a (degenerate) flat metric on S, and ω is invariant under translations when it is written in local coordinates. Let $\mathcal{H}_g$ be the moduli space of unit-area translation surfaces of genus $g \geq 2$. There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on translation surfaces and on their moduli, and the Teichmüller geodesic flow arises as the one-parameter diagonal subgroup of $\mathrm{SL}_2(\mathbb{R})$. It is shown in breakthrough work of Eskin-Mirzakhani and Eskin-Mirzakhani-Mohammadi [7, 8] that for any $\omega \in \mathcal{H}_g$, the closure X of $\mathrm{SL}_2(\mathbb{R}) \cdot \omega$ is an affine invariant submanifold, and supports an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure ν.

1.1.1 Oscillations for fixed surfaces

Let $\pi : \mathbf{H} \to X$ be the absolute (real) Hodge bundle over an $\mathrm{SL}_2(\mathbb{R})$ orbit closure X, whose $2g$-dimensional fiber over each point in X is $H^1(S, \mathbb{R})$. Let ν be an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure on X. For $g \in \mathrm{SL}_2(\mathbb{R})$, the Kontsevich-Zorich cocycle g_* is the lift of the action of g to $\mathbf{H}$, obtained by parallel transport with respect to the Gauss-Manin connection. Moreover, g_* acts symplectically since it preserves the intersection form on $H^1(S, \mathbb{R})$.

Fix a norm $\|\cdot\|_{\pi(\cdot)}$ on $\mathbf{H}$. Define $\sigma : \mathrm{SL}_2(\mathbb{R}) \times \mathbf{H} \to \mathbb{R}$ by

$$\sigma(g, \mathbf{v}) = \frac{\|g_*\mathbf{v}\|_{g\pi(\mathbf{v})}}{\|\mathbf{v}\|_{\pi(\mathbf{v})}}$$

Let $\mathbf{V}$ be a ν-strongly irreducible $\mathrm{SL}_2(\mathbb{R})$-invariant subbundle in the symplectic orthogonal of the tautological subbundle, which is defined and is continuous on X [9]. For each $\omega \in X$ and $\mathbf{v}_\omega \neq 0$ in $\mathbf{V}_\omega$, and for a.e. $r_\theta \in \mathrm{SO}_2(\mathbb{R})$, it is a consequence of a theorem of Chaika-Eskin [10] that

$$\lim_{t\to\infty} \frac{1}{t} \log \sigma(g_t r_\theta, \mathbf{v}_\omega) = \lambda \tag{1.1.1.1}$$

where $\lambda = \lambda(\nu)$ is the top Lyapunov exponent of the restriction of the Kontsevich-Zorich cocycle to $\mathbf{V}$.

Let h_t be the Teichmüller horocycle flow and let $\mathrm{T}h_t := (h_t)_*$ be its lift to the projectivized Hodge bundle $\mathbb{P}(\mathbf{H})$. Suppose that one is interested in studying the probability measures on $\mathbb{P}(\mathbf{H})$ that are invariant under $\mathrm{T}h_t$. Motivated by the work of Bainbridge-Smillie-Weiss [11], Forni posed the following question to us:

Problem 1.1.1 (G. Forni)**.** Let $\hat{\mu}$ be a $\mathrm{T}h_t$-invariant probability measure supported on the projectivized bundle $\mathbb{P}(\mathbf{H})$. Is it true that the push-forward measure μ under the projection map to the moduli space must be supported on an orbit closure with completely degenerate Kontsevich-Zorich exponents?

In the symplectic orthogonal of the tautological subbundle, it is known that orbit closure with completely degenerate Kontsevich-Zorich exponents exist in the strata $\mathcal{H}(1,1,1,1)$

and $\mathcal{H}^{\text{even}}(2,2,2)$ [12, 13], and are referred to in the literature as *Eierlegende Wollmilchsau* and *Ornithorynque*. It is also known that the cocycle acts by isometries in these two orbit closures, and so by a Krylov-Bogoliubov construction, one can construct $\mathrm{T}h_t$-invariant probability measures that are non-trivial (that is, not supported on the zero-section). One can then ask whether these are the only possible examples that arise. In all other situations, we expect that oscillations of the norm of the cohomology classes along large circles, when they exist, prevent the existence of non-trivial $\mathrm{T}h_t$-invariant probability measures in general.

In Chapter 2 of this thesis, we give partial evidence (in the affirmative) towards Forni's question, and present a mechanism for producing oscillations along the lift of the Teichmüller geodesic flow to the (real) Hodge bundle, as the basepoint surface is rotated. We apply our methods to strata with varying Lyapunov exponents. More precisely, we say that an orbit closure X has a varying Lyapunov phenomenon if there exists an affine invariant submanifold of X that supports an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant measure ν' and such that $\lambda(\nu') \neq \lambda(\nu)$, where $\lambda(\nu)$ is as in 1.1.1.1.

For such an X, we show that the cocycle cannot be normalized by a function that is independent of θ, for a full measure set of angles.

More precisely, we show

Theorem 1.1.2. *For any ω in X so that $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot \omega} = X$ with $\lambda = \lambda(\nu) > 0$, and such that X has a varying Lyapunov phenomenon, and for any function $f(t)$, the set*

$$\left\{ r_\theta \in \mathrm{SO}_2(\mathbb{R}) \ : \ \lim_{t\to\infty} \frac{\sigma(g_t r_\theta, \mathbf{v}_\omega)}{f(t)} \text{ converges to a non-zero number} \right\}$$

has zero measure with respect to the Haar measure on $\mathrm{SO}_2(\mathbb{R})$.

The tools used in the proof include quantitative recurrence results of Eskin-Mirzakhani-Mohammadi [8] and Hodge norm estimates [4].

It is a fact due to Chen-Möller [6] that the sum of Lyapunov exponents is varying for eight connected components of strata in genus 4, so that the varying Lyapunov phenomena holds for those connected components. Therefore, for any surface with full orbit closure, and for any Lagrangian subspace in the Hodge bundle, the conclusion of our theorem holds for those connected components of strata in genus 4 which exhibit varying Lyapunov phenomena.

It is natural then to propose the following problem

Problem 1.1.3. Beyond varying strata, find mechanisms for producing oscillations in any orbit closure that supports an $\mathrm{SL}_2(\mathbb{R})$-invariant measure ν with $\lambda(\nu) \neq 0$.

Finally, recall that Eskin–Mirzakhani–Mohammadi prove that $1/T \int_0^T (g_t)_*(\nu)$ converges to the unique $\mathrm{SL}_2(\mathbb{R})$-invariant measure for any horocycle invariant measure ν (and, by work of Forni, the averaging can be removed so that $(g_t)_*(\nu)$ converges to the the unique $\mathrm{SL}_2(\mathbb{R})$-invariant measure outside a set of times of density zero). The oscillation mechanism presented in Chapter 2 has recently been greatly developed and refined by Chaika, Khalil, and Smillie in their work on limit measures of Teichmüller horocyclic arcs (cf. [14, Theorem 4.1]). In fact, their work concerns the pushforward of measures supported on periodic orbits of the horocycle flow by the geodesic flow. They prove that for sequences of the form $(g_{t_k})_*(\nu_{t_k})$, the limit can fail to be ergodic. However, the original motivating question is whether $(g_t)_*(\nu)$, the pushforward of a *fixed* horocycle flow invariant measure ν by the geodesic flow, also fails to be ergodic in the limit. This question is still open, and we expect that the oscillations of the Kontsevich-Zorich cocycle give a potential obstruction to the conclusion that the limit exists.

1.1.2 Central limit theorem for generic surfaces

Suppose, in contrast to the setting described in the previous section, that we allow the surface to be random with respect to the measure ν. Then some quantitative information can be extracted.

In Chapter 3 of this thesis, and following the potential-theoretic approach to studying Lyapunov exponents in Teichmüller dynamics (pioneered by Kontsevich-Zorich [15] and further developed by Forni [4]), and inspired by the work of Le Jan [16], we study the trajectories of the (foliated) hyperbolic Brownian motion, prove a central limit theorem for the lift of these trajectories to the Hodge bundle, and deduce a central limit theorem for the Kontsevich-Zorich cocycle. Our approach is based on approximating hyperbolic geodesics by trajectories of the hyperbolic Brownian motion.

Our result can be contrasted with those that exist in the literature of products of iid random matrices. Concerning such a "noncommutative CLT", see for example the results of Le Page in [17] and Benoist and Quint in [18]. It is then also of interest to show that such noncommutative CLTs also exist for deterministic $\mathrm{SL}_n(\mathbb{R})$-valued cocycles, of which the Kontsevich-Zorich cocycle is certainly an example.

More precisely, let $\pi : \mathbb{P}(\mathbf{H}) \to X$ be the projectivized absolute (real) Hodge bundle over an $\mathrm{SL}_2(\mathbb{R})$ orbit closure X, whose fiber over each point in X is $H^1(S, \mathbb{R})$, and with ν

an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure on X. For $g \in \mathrm{SL}_2(\mathbb{R})$, the Kontsevich-Zorich cocycle $(g)_*$ is the lift of the action of g to $\mathbb{P}(\mathbf{H})$, obtained by parallel transport with respect to the Gauss-Manin connection. Furthermore, $(g)_*$ acts symplectically since it preserves the intersection form on $H^1(S, \mathbb{R})$.

For our purposes, we will in fact be concerned with the top exterior power of the Hodge bundle that is symplectic orthogonal to the tautological subbundle (spanned by $[\mathrm{Re}\,\omega]$ and $[\mathrm{Im}\,\omega]$), and we continue to call this subbundle $\mathbb{P}(\mathbf{H})$. This bundle supports an $\mathrm{SO}_2(\mathbb{R})$-invariant probability measure $\hat{\nu}$ such that, for ν-a.e ω, the conditional measure on $\mathbb{P}(\mathbf{H}_\omega)$ is the Haar measure. An Euclidean structure is in fact given by the Hodge norm (see 3.7.5 for the definition), and which we use in the sequel.

Therefore, we fix the $\mathrm{SO}_2(\mathbb{R})$-invariant Hodge norm $\|\cdot\|_{\pi(\cdot)}$ on $\mathbf{H}$. Define σ : $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{P}(\mathbf{H}) \to \mathbb{R}$ by

$$\sigma(g, \mathbf{v}) = \log \frac{\|(g)_* \mathbf{v}\|_{g\pi(\mathbf{v})}}{\|\mathbf{v}\|_{\pi(\mathbf{v})}}$$

For $\omega \in X$, let $\mathbf{v}_\omega$ in $\mathbb{P}(\mathbf{H}_\omega)$ be the projectivization of any Lagrangian subspace (of dimension $g-1$) in $\mathbb{P}(\mathbf{H})$. For $\hat{\nu}$-a.e. $\mathbf{v} = (\omega, \mathbf{v}_\omega)$, it is a consequence of the multiplicative ergodic theorem that

$$\lim_{T\to\infty} \frac{\sigma(g_T, \mathbf{v})}{T} = \sum_{i=2}^{g} \lambda_i$$

where, together with $\lambda_1 = 1$, λ_i are the top g Lyapunov exponents of the Kontsevich-Zorich cocycle, and where $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. The top g exponents determine the entire Lyapunov spectrum by symplecticity. Note that $\lambda_2 = 0$ if and only if $\sum_{i=2}^{g} \lambda_i = 0$, since the exponents are ordered so that $\lambda_2 \geq \cdots \geq \lambda_g \geq 0$.

The main result of Chapter 3 is the following:

Theorem 1.1.4. *There exists $\Phi_{g_\infty} \geq 0$ such that*

$$\lim_{T\to\infty} \hat{\nu}\left(\left\{\mathbf{v} \in \mathbb{P}(\mathbf{H}) \ : \ a \leq \frac{1}{\sqrt{T}}(\sigma(g_T, \mathbf{v}) - T\lambda) \leq b\right\}\right) = \frac{1}{\sqrt{2\pi\Phi_{g_\infty}}} \int_a^b \exp(-x^2/\Phi_{g_\infty})dx.$$

Remark 1.1.5. *The statement also holds for when $\Phi_{g_\infty} = 0$, in that the resulting distribution is a delta distribution.*

To prove this, we will first work with the hyperbolic Brownian motion, which is the diffusion process generated by the foliated hyperbolic Laplacian. Let ρ be a (foliated) hyperbolic Brownian motion trajectory starting at a (generic) basepoint $\omega \in X$, defined almost everywhere with respect to a probability measure $\mathbb{P}_\omega$ on the space of such trajectories W_ω. This process is in fact defined on $X^* = \mathrm{SO}_2(\mathbb{R})\backslash X$. Moreover, ρ can be lifted to $\mathrm{SL}_2(\mathbb{R})$, and is moreover defined by taking the outward radial unit tangent vector at all points. We continue to refer to the lifted path as ρ by abuse of notation. Additionally, the space X gives rise to a product space $X^W := X \otimes W$ whose fiber over each point ω in X is W_ω, and which also supports a measure $\nu_{\mathbb{P}} := \nu \otimes \mathbb{P}$, whose conditional measure over a point ω is $\mathbb{P}_\omega$. We can thus similarly define the product W-Hodge bundle $\mathbb{P}^W(\mathbf{H})$, whose fiber over each point (ω, ρ) in X^W is $\mathbf{H}_\omega$. A pair $(\rho, \mathbf{v}) \in \mathbb{P}^W(\mathbf{H})$ is thus defined to be the lift of the path ρ (starting at ω) to $\mathbb{P}^W(\mathbf{H})$, obtained by parallel transport with respect to the Gauss-Manin connection. This in turn would also give rise to a measure $\hat{\nu}_{\mathbb{P}} := \hat{\nu} \otimes \mathbb{P}$ whose conditional measure over a point $\mathbf{v}$ is $\mathbb{P}_\omega$. We therefore also have

Theorem 1.1.6. *There exists $\Phi_{\rho_\infty} \geq 0$ such that*

$$\lim_{T\to\infty} \hat{\nu}_{\mathbb{P}} \left(\left\{ \rho_{\mathbf{v}} \in \mathbb{P}^W(\mathbf{H}) \ : \ a \leq \frac{1}{\sqrt{T}}(\sigma(\rho_T, \mathbf{v}) - T\lambda) \leq b \right\} \right) = \frac{1}{\sqrt{2\pi\Phi_{\rho_\infty}}} \int_a^b \exp(-x^2/\Phi_{\rho_\infty}) dx.$$

Moreover, if $\lambda_2 > 0$, then $\Phi_{\rho_\infty} > 0$.

Remark 1.1.7. *Observe that for $g = 2$, our two results reduce to ones that concern the second Lyapunov exponent λ_2.*

Some ingredients of our proof include

- results of Avila-Gouëzel-Yoccoz [19] and Avila-Gouëzel [20] on the spectral gap of the leafwise hyperbolic Laplacian to show existence of a solution of the leafwise Poisson's equation (see Appendix .1),
- elementary stochastic calculus to extract and control the necessary oscillations,
- and an asymptotic estimate due to Ancona [21] to relate the geodesic flow with the Brownian motion.

Some of the natural problems that come up from our work include

Problem 1.1.8. Show that the variance Φ_{g_∞} is positive when $\lambda_2 > 0$.

In another direction, observe that our result already gives a CLT for λ_2 in genus 2. It is natural then to also propose

Problem 1.1.9. For $g > 2$, remove the $(g-1)$-dimensional constraint on the isotropic subspace $\mathbf{v}_\omega$ in Theorem 1.1.4.

This would give a CLT for each individual Lyapunov exponent beyond the genus 2 case.

To describe a potential application, let us recall the following

Theorem 1.1.10 (Kesten)**.** *[22, 23] Let* $0 < r < 1$*, and let*

$$D_N(r,x,\alpha) = \sum_{n=0}^{N-1} \chi_{[0,r]}(x+n\alpha) - Nr.$$

There is a number $\rho = \rho(r)$ *such that if* (x,α) *is uniformly distributed on* $\mathbb{T}^2$ *then* $\frac{D_N}{\rho \log N}$ *converges to a standard Cauchy distribution. Moreover,* ρ *is independent of* r *if* r *is irrational, and has a non-trivial dependence on* r *otherwise.*

See also the work of Dolgopyat-Fayad in [24, 25] for higher-dimensional generalizations of this theorem, and the work of Bufetov on limit theorems for translation flows [26]. In view of this, it is a natural then to ask whether an analogue of that theorem could exist for generalized rotations, i.e. interval exchange transformations, and also for straight-line flows on translation surfaces. That is,

Problem 1.1.11 (C. Ulcigrai)**.** Does an analogue of Kesten's theorem hold for generalized rotations and translation flows?

The expectation is that a nondegenerate central limit theorem for the Kontsevich-Zorich cocycle, one that holds for *individual* Lyapunov exponents, would give a negative answer to this question.

Chapter 2: Oscillations and the Konstevich-Zorich Cocycle

2.1 Abstract

We present a mechanism for producing oscillations along the lift of the Teichmüller geodesic flow to the (real) Hodge bundle, as the basepoint surface is deformed by a unipotent element of $\mathrm{SL}_2(\mathbb{R})$. We apply our methods to all connected components of strata which exhibit a varying phenomenon for the sum of non-negative Lyapunov exponents. In genus 4, by work of Chen-Möller, 8 connected components of strata exhibit varying Lyapunov exponents, and so we apply our methods to those connected components that are shown to be varying by their work.

2.2 Introduction

Let $\pi : \mathbf{H} \to X$ be the absolute (real) Hodge bundle over an $\mathrm{SL}_2(\mathbb{R})$ orbit closure X, whose $2g$-dimensional fiber over each point in X is $H^1(S, \mathbb{R})$. Let ν be an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure on X. For $g \in \mathrm{SL}_2(\mathbb{R})$, the Kontsevich-Zorich cocycle g_* is the lift of the action of g to $\mathbf{H}$, obtained by parallel transport with respect to the Gauss-Manin connection. Moreover, g_* acts symplectically since it preserves the intersection form on $H^1(S, \mathbb{R})$.

Let h_t be the Teichmüller horocycle flow and let $\mathrm{T}h_t := (h_t)_*$ be its lift to the projectivized Hodge bundle $\mathbb{P}(\mathbf{H})$. Suppose that one is interested in studying the probability measures on $\mathbb{P}(\mathbf{H})$ that are invariant under $\mathrm{T}h_t$. Motivated by the work of Bainbridge-Smillie-Weiss [11], Forni posed the following question:

Question 2.2.1 (G. Forni)**.** Let $\hat{\mu}$ be a $\mathrm{T}h_t$-invariant probability measure supported on the projectivized bundle $\mathbb{P}(\mathbf{H})$. Is it true that the push-forward measure μ under the projection map π to the moduli space must be supported on an orbit closure with completely degenerate Kontsevich-Zorich exponents?

In the symplectic orthogonal of the tautological subbundle, it is known that orbit closures with completely degenerate Kontsevich-Zorich exponents exist in the strata $\mathcal{H}(1,1,1,1)$ and $\mathcal{H}^{\text{even}}(2,2,2)$ [12, 13, 27], and are referred to in the literature as *Eierlegende Wollmilchsau* and *Ornithorynque*. That these are the only orbit closures with completely degenerate Kontsevich-Zorich exponents follows from the works [28, 29, 30, 31]. It is also known that the cocycle acts by isometries in these two orbit closures, and so by a Krylov-Bogoliubov construction, one can construct $\mathrm{T}h_t$-invariant probability measures that are non-trivial (that is, not supported on the zero-section). One can then ask whether these are the only possible examples that arise. In all other situations, we expect that oscillations of the norm of the cohomology classes along large circles, when they exist, prevent the existence of non-trivial $\mathrm{T}h_t$-invariant probability measures in general.

In this paper, we give partial evidence (in the affirmative) towards Forni's question, and present a mechanism for producing oscillations along the lift of the Teichmüller geodesic flow to the (real) Hodge bundle, as the basepoint surface is rotated.

More precisely, fix a norm $\|\cdot\|_{\pi(\cdot)}$ on $\mathbf{H}$. Define $\sigma : \mathrm{SL}_2(\mathbb{R}) \times \mathbf{H} \to \mathbb{R}$ by

$$\sigma(g, \mathbf{v}) = \frac{\|g_* \mathbf{v}\|_{g\pi(\mathbf{v})}}{\|\mathbf{v}\|_{\pi(\mathbf{v})}}$$

Let $\mathbf{V}$ be a ν-strongly irreducible $\mathrm{SL}_2(\mathbb{R})$-invariant subbundle in the symplectic orthogonal of the tautological subbundle, which is defined and is continuous on X [9].

For each $\omega \in X$ with full orbit closure, and $\mathbf{v}_\omega \neq 0$ in $\mathbf{V}_\omega$, and for a.e. $r_\theta \in \mathrm{SO}_2(\mathbb{R})$, it is a consequence of a theorem of Chaika-Eskin [10] that

$$\lim_{t\to\infty} \frac{1}{t} \log \sigma(g_t r_\theta, \mathbf{v}_\omega) = \lambda$$

where $\lambda = \lambda(\nu)$ is the top Lyapunov exponent of the restriction of the Kontsevich-Zorich cocycle to $\mathbf{V}$. See [32] for a refinement of this result to a positive Hausdorff codimension set of angles.

We say that an orbit closure X has a **varying Lyapunov phenomenon** if there exists an affine invariant submanifold of X that supports an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant measure ν' and such that $\lambda(\nu') \neq \lambda(\nu)$. For such an X, we show that the cocycle cannot be normalized by a function that is independent of θ, for a full measure set of angles.

More precisely, letting $h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, $\bar{h}_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$, $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ and $r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, we show

Theorem 2.2.2. *For any ω in X so that $\overline{\mathrm{SL}_2(\mathbb{R}) \cdot \omega} = X$ with $\lambda = \lambda(\nu) > 0$, and such* *that X has a varying Lyapunov phenomenon, and for any function $f(t)$, the set*

$$\left\{ r_\theta \in \mathrm{SO}_2(\mathbb{R}) \ : \ \lim_{t\to\infty} \frac{\sigma(g_t r_\theta, \mathbf{v}_\omega)}{f(t)} \text{ converges to a non-zero number} \right\}$$

has zero measure with respect to the Haar measure on $\mathrm{SO}_2(\mathbb{R})$.

We show that Theorem 2.2.2 follows from its horocyclic counterpart

Theorem 2.2.3. *Under the hypothesis in Theorem 2.2.2, the set*

$$\left\{ s \in [-1, 1] \ : \ \lim_{t\to\infty} \frac{\sigma(g_t h_s, \mathbf{v}_\omega)}{f(t)} \text{ converges to a non-zero number} \right\}$$

has zero measure with respect to the Lebesgue measure on $[-1, 1]$.

2.2.1 Sketch of the Proof

The idea of the proof is to construct two regimes, one in which the Kontsevich-Zorich cocycle grows as expected, and another where the behavior is atypical.

The expected behavior is an input that comes from the work of Chaika-Eskin [10], where it is shown that one can construct a large open set in which the cocycle grows as expected in finite time (i.e. exponentially with rate λ up to some additive error), and whose set of bad futures is small.

The atypical behavior is extracted from carefully defined tubular neighborhoods of a submanifold of X whose (second) largest Lyapunov exponent is varying. To conclude the argument, we first use quantitive recurrence results from Eskin-Mirzakhani-Mohammadi [8] to control returns to the typical and atypical neighborhoods, then apply a Lebesgue density argument to deduce that the limit cannot exist for a full measure set of directions.

2.2.2 Applications

For $\omega \in X$, let $\mathbf{v}_\omega$ in $\mathbf{H}_\omega$ be any isotropic subspace of dimension $g-1$ in the symplectic orthogonal of the tautological subspace, span$\{[\text{Re}\ \omega], [\text{Im}\ \omega]\}$, of the Hodge bundle $\mathbf{H}$. For each $\omega \in X$ with full orbit closure, and for a.e. $r_\theta \in \mathrm{SO}_2(\mathbb{R})$, it is also a consequence of the same theorem of Chaika-Eskin [10] referred to above that

$$\lim_{t\to\infty} \frac{1}{t} \log \sigma(g_t, \mathbf{v}_\omega) = \sum_{i=2}^{g} \lambda_i$$

where, together with $\lambda_1 = 1$, λ_i are the top g Lyapunov exponents of the Kontsevich-Zorich cocycle. The top g exponents determine the entire Lyapunov spectrum by symplecticity. The work of Chen-Möller [6] demonstrates, in part, the following result:

Theorem 2.2.4. *[6, Theorem 1.2] There are eight connected components of strata in genus 4 where the sum of Lyapunov exponents is varying.*

Thus the conclusions of Theorem 2.2.2 and 2.2.3 hold for any surface with full orbit closure in eight connected components of strata in genus 4.

2.2.3 Related results

In [33], Dolgopyat-Fayad-Vinogradov prove a central limit theorem for the time integral of sufficiently regular zero-average observables of the pushforward of a small horocyclic arc by the geodesic flow, as the basepoint varies generically with respect to an ergodic P-invariant measure, where P is the upper triangular subgroup of $\mathrm{SL}_2(\mathbb{R})$ (their results are in fact much more general, cf. [33, Theorem 7.1], but we present their theorem in the $\mathrm{SL}_2(\mathbb{R})$ setting for simplicity). While their results are not immediately applicable to the Kontsevich-Zorich cocycle, it shows that Theorem 2.2.3, which is, in our specialized setting, a qualitative analogue of their results, can likely be strengthened quantitatively, if one allows the surface to vary generically in an orbit closure (so this would be a tradeoff, since our result is true for any fixed basepoint, thanks to the work of Eskin-Mirzakhani-Mohammadi). That a central limit theorem holds for the Kontsevich-Zorich cocycle as a basepoint varies generically is the subject of [34], and the approach in that paper uses completely different tools (the Brownian motion).

Recently, and after the completion of this paper, the oscillation mechanism presented here has been greatly developed and refined by Chaika, Khalil, and Smillie in their work on limit measures of Teichmüller horocyclic arcs (cf. [14, Theorem 4.1]).

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2.3 Preliminaries

2.3.1 Translation surfaces

Let S be a Riemann surface of genus $g \geq 2$, and ω a holomorphic 1-form on S. The pair (S, ω) is said to be a translation surface, since ω gives a (degenerate) flat metric on S, and ω is invariant under translations when it is written in local coordinates. The zero set Σ of ω characterizes the singularity set of the conical metric. The area of a translation surface is given by $\int_S \omega \wedge \overline{\omega}$. We refer to the pair (S, ω) as just ω.

2.3.2 Moduli Space

Let $\mathcal{TH}_g$ be the Teichmüller space of unit-area translation surfaces of genus $g \geq 2$, and let $\mathcal{H}_g = \mathcal{TH}_g/\mathrm{Mod}_g$ be the corresponding moduli space, where Mod_g is the mapping class group. The space $\mathcal{H}_g$ is partitioned into strata $\mathcal{H}(\kappa) = \mathcal{H}(\kappa_1, \ldots, \kappa_n)$, which consist of unit-area translation surfaces whose singularities have cone angle $2\pi\kappa_i$, and $\sum \kappa_i = 2g - 2$. One can also define local period coordinates in a stratum, where all changes of coordinates are given by affine maps.

2.3.3 $\mathrm{SL}_2(\mathbb{R})$ action

There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on translation surfaces and on their moduli. It is shown in [7, 8] that for any $\omega \in \mathcal{H}(\kappa)$, the closure X of $\mathrm{SL}_2(\mathbb{R}) \cdot \omega$ is an affine invariant submanifold, and supports an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure ν.

2.3.4 Hodge inner product

Given two holomorphic 1-forms ω_1, ω_2 in $\Omega(S)$, where $\Omega(S)$ is the vector space of holomorphic 1-forms on S, the Hodge inner product is defined to be

$$\langle \omega_1, \omega_2 \rangle := \frac{i}{2} \int_S \omega_1 \wedge \overline{\omega_2}$$

Moreover, the Hodge representation theorem implies that for any given cohomology class $c \in H^1(S, \mathbb{R})$, there is a unique holomorphic 1-form $h(c) \in \Omega(S)$, such that $c = [\text{Re } h(c)]$. We define the Hodge inner product for two real cohomology classes $c_1, c_2 \in H^1(S, \mathbb{R})$ as

$$(c_1, c_2)_\omega := \langle h(c_1), h(c_2) \rangle_\omega$$

For c in the symplectic orthogonal of $[\omega]$, it also follows from the work of Forni [4] (see also [35, Corollary 2.1] and [36, Corollary 30]) that

$$\left| \frac{d}{dt} \log \|c\|_{g_t \omega} \right| < 1 \tag{2.3.4.1}$$

2.3.5 Expected behavior

For each of the ambient manifold X and a submanifold X', we will need to construct open sets where the cocycle grows as expected up to some additive error, and whose set of bad futures is small. This is implemented in [10] for random walks, and is adapted to the deterministic case in [32, Corollary 7.6]. The main point is that random walks track geodesics up to an error that is sublinear in hyperbolic distance, and the rest of the adaptation follows by standard arguments and [10, Lemma 2.11].

To that end, let $\epsilon > 0$ and $L \in \mathbb{N}$. Let $E_{good}(\epsilon, L)$ be the subset of X such that for any $\omega \in E_{good}(\epsilon, L)$ and any $\mathbf{v}_\omega \in \mathbf{V}_\omega$, there is a subset $H(\mathbf{v}_\omega)$ of $[-1, 1]$ so that

$$\mu(H(\mathbf{v}_\omega)) \geq 2 - \epsilon$$

and such that for all $s \in H(\mathbf{v}_\omega)$

$$\lambda - \epsilon < \frac{\log \sigma(g_L h_s, \mathbf{v}_\omega)}{L} < \lambda + \epsilon \tag{2.3.5.1}$$

Corollary 2.3.1. *[10, Lemma 2.11] For any $\epsilon > 0$ and $\delta > 0$, there exists $L_0 > 0$ such that for all $L > L_0$, we have that $\nu(E_{good}(\epsilon, L)) > 1 - \delta$.*

Similarly, for the r_θ action on a submanifold X' of X that supports some measure ν' with $\lambda' = \lambda(\nu')$, set $\epsilon' > 0$ and $L \in \mathbb{N}$. Let $E'_{good}(\epsilon', L)$ be the subset of X' such that for any $\omega \in E'_{good}(\epsilon', L)$ and any $\mathbf{v}_\omega \in \mathbf{V}_\omega$, there is a subset $H'(\mathbf{v}_\omega)$ of $[-1, 1]$ so that

$$\mu(H'(\mathbf{v}_\omega)) \geq 2 - \epsilon$$

and such that for all $\theta \in H'(\mathbf{v}_\omega)$

$$\lambda' - \epsilon' < \frac{\log \sigma(g_L r_\theta, \mathbf{v}_\omega)}{L} < \lambda' + \epsilon' \tag{2.3.5.2}$$

Corollary 2.3.2. *[10, Lemma 2.11] For any $\epsilon' > 0$ and $\delta' > 0$, there exists $L_0 > 0$ such* *that for all $L > L_0$, we have that $\nu'(E'_{good}(\epsilon', L)) > 1 - \delta'$.*

We refer to [32, Lemma 7.5, Corollary 7.6] for the adapted proofs.

2.3.6 Exceptional behavior

Since X is assumed to have varying Lyapunov phenomenon, there exists an affine invariant submanifold X' of X that supports an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant measure ν', and such that $\lambda(\nu') \neq \lambda(\nu)$. Set $\lambda' = \lambda(\nu')$. In this section, we show that in a neighborhood of the manifold X', the exceptional behavior is also present in some open subset of the ambient orbit closure X up to some prescribed time L.

For $\beta > 0$, let X_β be the β-thick part of X, which is the subset of X such that for all $\omega \in X$, saddle connections have ω-length at least β. For the pseudo-metric d_{Teich} on X, define the dynamical metric

$$d^n_{Teich}(x, y) := \sup_{i\in[0,n]} d_{Teich}(g_n x, g_n y).$$

Let $E'_{good}(\epsilon', L) \subset X'$ be as in Corollary 2.3.2. By the tubular neighborhood theorem, for any $L > 0$, there exists a $\beta > 0$ such that

$$N_{L,\epsilon'}(X') = \left\{\omega \in X_\beta \ : \ d^L_{Teich}(\omega, E'_{good}(\epsilon', L)) < \frac{1}{L}\right\}$$

is a non-empty open set that is contained in some thick part X_β of X.

For any $\omega \in N_{L,\epsilon'}(X')$, it follows by construction that there exists $\omega' \in X'$ such that $d_{Teich}(\omega, \omega') < 1/L$ and $d_{Teich}(g_t\omega, g_t\omega') < 1/L$ for any $t \leq L$. Let $\theta_1 \in H'(\mathbf{v}_{\omega'})$ be an admissible direction as in Corollary 2.3.2. For all $\mathbf{v}_\omega$ in the symplectic orthogonal of $[\omega]$, we have by 2.3.4.1 and the cocycle property that

$$\begin{aligned}
\log \sigma(g_t, \mathbf{v}_\omega) \leq\ & d_{Teich}(\omega, \omega') + \log \sigma(r_{\theta_1}, \mathbf{v}_{\omega'}) + \log \sigma(g_t, r_{\theta_1}\mathbf{v}_{\omega'}) \\
& \log \sigma(r_{\theta_2}, g_t r_{\theta_1}\mathbf{v}_{\omega'}) + d_{Teich}(g_t\omega, g_t\omega') \\
& = d_{Teich}(\omega, \omega') + \log \sigma(g_t, r_{\theta_1}\mathbf{v}_{\omega'}) + d_{Teich}(g_t\omega, g_t\omega')
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\log \sigma(g_t, \mathbf{v}_\omega) &\leq d_{Teich}(\omega, \omega') + \log \sigma(g_t, r_{\theta_1}\mathbf{v}_{\omega'}) + d_{Teich}(g_t\omega, g_t\omega') \\
&< \frac{1}{L} + (\lambda' + \epsilon')t + \frac{1}{L} \\
&< 2 + (\lambda' + \epsilon')t
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
\log \sigma(g_t, \mathbf{v}_\omega) &\geq -d_{Teich}(\omega, \omega') + \log \sigma(g_t, r_{\theta_1}\mathbf{v}_{\omega'}) - d_{Teich}(g_t\omega, g_t\omega') \\
&> -\frac{1}{L} + (\lambda' - \epsilon')t - \frac{1}{L} \\
&> -2 + (\lambda' - \epsilon')t
\end{aligned}$$

for all $t \leq L$. This gives in particular that

$$-2 + (\lambda' - \epsilon')t \leq \log \sigma(g_t, \mathbf{v}_\omega) \leq 2 + (\lambda' + \epsilon')t \tag{2.3.6.1}$$

for all $t \leq L$ and $\omega \in N_{L,\epsilon'}(X')$.

2.3.7 Quantitative recurrence

Recall that $X = \overline{\mathrm{SL}_2(\mathbb{R})\omega}$. The major ingredient in our work is

Theorem 2.3.3. *[8, Theorem 2.10] For $f \in C_c(X)$, any $\epsilon > 0$, and any interval $I \subset \mathbb{R}$, there exists $T_0 > 0$ such that for all $T > T_0$, we have*

$$\left| \frac{1}{T} \int_0^T \frac{1}{|I|} \int_I f(g_t h_s \omega) ds dt - \int_X f d\nu \right| < \epsilon$$

The following proposition then follows from Theorem 2.3.3, whose analogue for the r_θ action appears in the work of Chaika-Lindsey [37, Proposition 8].

Proposition 2.3.4. *[37] Let U_1, U_2 be any open sets in X, and let Z be any interval. For any $\epsilon > 0$, there exists arbitrarily large times $T > 0$ such that for each $i \in \{1, 2\}$,*

$$|\{s \in Z \ : \ g_T h_s \omega \in U_i\}| \geq |Z|(\nu(U_i) - \epsilon)$$

2.4 Proofs of Theorems 2.2.2 and 2.2.3

2.4.1 Theorem 2.2.3 for horocyclic arcs

Recall we want to show that the set

$$E = \left\{ s \in [-1, 1] \ : \ \lim_{t \to \infty} \frac{\sigma(g_t h_s, \mathbf{v}_\omega)}{f(t)} \text{ converges to a non-zero number} \right\}$$

has zero measure with respect to the Lebesgue measure on $[-1, 1]$.

We argue by contradiction. Assume that the set E has positive measure. Restrict E to a smaller set (which we continue to call E) where we have uniform convergence to the limit C_s (by Egorov). More precisely, by uniformity, we have that for any $\eta > 0$, there exists t_η such that

$$\left|\frac{\sigma(g_t h_s, \mathbf{v}_\omega)}{f(t)} - C_s\right| < \eta$$

for all $s \in E$ for all $t > t_\eta$. This implies that for $s_1, s_2 \in E$,

$$\frac{C_{s_1} - \eta}{C_{s_2} + \eta} < \frac{\sigma(g_t h_{s_1}, \mathbf{v}_\omega)}{\sigma(g_t h_{s_2}, \mathbf{v}_\omega)} < \frac{\eta + C_{s_1}}{C_{s_2} - \eta}$$

Observe that C_s can be made continuous by applying Luzin's theorem and further restricting E to a smaller set (which we continue to call E), so that C_s is bounded over $s \in [-1, 1]$. Therefore, we have that the limits C_s are bounded from above and below by C and c, respectively. This gives us that for all $s_1, s_2 \in E(\eta)$, and all $t > t_\eta$.

$$\frac{c - \eta}{C + \eta} < \frac{C_{s_1} - \eta}{C_{s_2} + \eta} < \frac{\sigma(g_t h_{s_1}, \mathbf{v}_\omega)}{\sigma(g_t h_{s_2}, \mathbf{v}_\omega)} < \frac{\eta + C_{s_1}}{C_{s_2} - \eta} \leq \frac{\eta + C}{c - \eta} \tag{2.4.1.1}$$

Pick $0 < \eta < c$, and let

$$K = K(\eta) := \frac{\eta + C}{c - \eta} \tag{2.4.1.2}$$

so that 2.4.1.1 reduces to

$$\frac{1}{K} < \frac{\sigma(g_t h_{s_1}, \mathbf{v}_\omega)}{\sigma(g_t h_{s_2}, \mathbf{v}_\omega)} < K \tag{2.4.1.3}$$

The contradiction is reached if we can show that there are arbitrary large times such that

$$\frac{\sigma(g_t h_{s_1}, \mathbf{v}_\omega)}{\sigma(g_t h_{s_2}, \mathbf{v}_\omega)} > K \text{ or } \frac{\sigma(g_t h_{s_1}, \mathbf{v}_\omega)}{\sigma(g_t h_{s_2}, \mathbf{v}_\omega)} < \frac{1}{K}$$

for pairs $(s_1, s_2) \in E \times E$.

Since we assume E has positive measure, there is an interval $Z = Z(\gamma)$ of some Lebesgue density point in E, so that

$$\frac{|E \cap Z|}{|Z|} > 1 - \gamma \tag{2.4.1.4}$$

for any $1 > \gamma > 0$. This is an important reduction, as it gives us access to Corollary 2.3.4, which requires an interval.

Note that although all pairs coming from $E \times E$ satisfy 2.4.1.3 for the sake of contradiction, this need not be the case for all pairs in $Z \times Z$. So we need to estimate an exceptional set of pairs, which is $Z \times (Z - E) \cup (Z - E) \times Z$. It is immediate from 2.4.1.4 that

$$|Z - E| \leq \gamma |Z| \tag{2.4.1.5}$$

From 2.4.1.5, it follows that

$$|Z \times (Z - E) \cup (Z - E) \times Z| \leq 2\gamma |Z|^2 \tag{2.4.1.6}$$

Pick ϵ and ϵ' such that $\epsilon + \epsilon' < |\lambda - \lambda'|$, and let $\delta = \delta' = \frac{1}{2}$. Then by an application of Corollary 2.3.1 on X (resp., Corollary 2.3.2 on X'), there exists $L_0(\epsilon, \delta)$ (resp., $L_0'(\epsilon', \delta')$)

such that the conclusion of the corollary is satisfied with $\nu(E_{good}(\epsilon, L)) > 1/2$ (resp., $\nu(E'_{good}(\epsilon', L)) > 1/2$). Pick

$$L > \max\left\{L_0, L'_0, \frac{2 + 2\log K}{\lambda - \lambda' - \epsilon - \epsilon'}, \frac{2 + 2\log K}{\lambda' - \lambda - \epsilon - \epsilon'}\right\}.$$

To simplify notation, set

$$U_1 = E_{good}(\epsilon, L)$$

$$U_2 = N_{L,\epsilon'}(E'_{good}(\epsilon', L)).$$

Let

$$A_1 = \{s \in Z \ : \ g_t h_s \omega \in U_1\} \tag{2.4.1.7}$$

$$A_2 = \{s \in Z \ : \ g_t h_s \omega \in U_2\} \tag{2.4.1.8}$$

To show that there are pairs coming from $A_1 \times A_2$ that intersect $E \times E$, it suffices to show by 2.4.1.6 that γ can be chosen so that

$$|A_1 \times A_2| > 2\gamma |Z|^2 \tag{2.4.1.9}$$

Now, to ensure that 2.4.1.9 is satisfied, we want to show

$$|A_1| > \sqrt{2\gamma}|Z| + \epsilon|A_1| \implies |A_1| > \sqrt{2\gamma}\frac{|Z|}{1-\epsilon}$$
$$|A_2| > \sqrt{2\gamma}|Z|$$

and we note that $|A_1|$ needs to exceed $\epsilon|A_1|$ since that is the proportion of bad futures in Corollary 2.3.1 applied on $E_{good}(\epsilon, L) \subset X$.

Choose $\gamma > 0$ such that

$$\frac{\nu(U_1)}{2} > \frac{\sqrt{2\gamma}}{1-\epsilon'} \tag{2.4.1.10}$$
$$\frac{\nu(U_2)}{2} > \sqrt{2\gamma} \tag{2.4.1.11}$$

Let $m = \min\{\nu(U_1), \nu(U_2)\}$. By Proposition 2.3.4 (with $\epsilon = m/2$), for each $i = \{1, 2\}$, there exists arbitrarily large $T > t_\eta$ such that

$$\frac{|A_i|}{|Z|} = \frac{1}{|Z|}|\{s \in Z \;:\; g_T h_s \omega \in U_i\}| > \nu(U_i) - \frac{1}{2}m \geq \frac{1}{2}\nu(U_i) \tag{2.4.1.12}$$

Define

$$\tau = \inf\{t_\eta \leq r \leq T : g_r h_{s_i}\omega \in U_i \text{ for } i = \{1, 2\}$$
$$\text{and for some } (s_1, s_2) \in (A_1 \times A_2) \cap (E \times E)\}$$

Suppose $\lambda > \lambda'$. If 2.4.1.3 is not violated at τ, then we have

$$\frac{\sigma(g_{\tau+L}h_{s_1}, \mathbf{v}_\omega)}{\sigma(g_{\tau+L}h_{s_2}, \mathbf{v}_\omega)} \geq \frac{\exp((\lambda-\epsilon)L)\sigma(g_\tau h_{s_1}, \mathbf{v}_\omega)}{\exp(2+(\lambda'+\epsilon')L)\sigma(g_\tau h_{s_2}, \mathbf{v}_\omega)} \geq \frac{\exp((\lambda-\epsilon)L)}{\exp(2+(\lambda'+\epsilon')L)}\frac{1}{K}$$

where we have applied the RHS of 2.3.6.1 in the first inequality. Observe then for the chosen L, we have that

$$\frac{\exp((\lambda-\epsilon)L)}{\exp(2+(\lambda'+\epsilon')L)}\frac{1}{K} > K$$

so that the RHS of 2.4.1.3 is violated, which is our contradiction since (s_1, s_2) belongs to $E \times E$.

Now suppose instead that $\lambda < \lambda'$. If 2.4.1.3 is not violated at τ, then we have

$$\frac{\sigma(g_{\tau+L}h_{s_1}, \mathbf{v}_\omega)}{\sigma(g_{\tau+L}h_{s_2}, \mathbf{v}_\omega)} \leq \frac{\exp((\lambda+\epsilon)L)\sigma(g_\tau h_{s_1}, \mathbf{v}_\omega)}{\exp(-2+(\lambda'-\epsilon')L)\sigma(g_\tau h_{s_2}, \mathbf{v}_\omega)} \leq \frac{\exp((\lambda+\epsilon)L)}{\exp(-2+(\lambda'-\epsilon')L)}K$$

where we have applied the LHS of 2.3.6.1 in the first inequality. Observe then for the chosen L, we have

$$\frac{\exp((\lambda+\epsilon)L)}{\exp(-2+(\lambda'-\epsilon')L)}K < \frac{1}{K}$$

so that the LHS of 2.4.1.3 is violated, which is our contradiction since (s_1, s_2) belongs to $E \times E$. $\square$

2.4.2 Theorem 2.2.2 for circle arcs

It is straightforward to see how Theorem 2.2.2 can be deduced from Theorem 2.2.3. Indeed, for any $\theta \neq \pm\pi/2$, we have

$$r_\theta = \bar{h}_{\tan\theta} g_{\log\cos\theta} h_{-\tan\theta}.$$

Since $g_t \bar{h}_{\tan\theta} = \bar{h}_{e^{-2t}\tan\theta} g_t$, we also have

$$g_t r_\theta = \bar{h}_{e^{-2t}\tan\theta} g_{t+\log\cos\theta} h_{-\tan\theta}.$$

So that s belongs to the set in Theorem 2.2.3 iff θ belongs to the set in Theorem 2.2.2. $\square$

3.5 Abstract

In this note, we show that a central limit theorem holds for the top exterior power of the Kontsevich-Zorich cocycle. In particular, we show that a central limit theorem holds for the lift of the (leafwise) hyperbolic Brownian motion to the Hodge bundle, and then show that a (possibly degenerate) central limit theorem holds for the lift of the Teichmüller geodesic flow to the same bundle. We show that the variance of the random cocycle is positive if the second top Lyapunov exponent of the cocycle is positive.

3.6 Introduction

Following the potential-theoretic approach to studying Lyapunov exponents in Teichmüller dynamic (pioneered by Kontsevich-Zorich [15] and further developed by Forni [4]), and inspired by the work of Le Jan [16], we study the trajectories of the (foliated) hyperbolic Brownian motion, prove a central limit theorem for the lift of these trajectories to the Hodge bundle, and deduce a central limit theorem for the top exterior power of the Kontsevich-Zorich cocycle.

More precisely, let $\pi : \mathbb{P}(\mathbf{H}) \to X$ be the projectivized absolute (real) Hodge bundle over an $\mathrm{SL}_2(\mathbb{R})$ orbit closure X, whose fiber over each point in X is $H^1(S, \mathbb{R})$, and with ν an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability measure on X. For $g \in \mathrm{SL}_2(\mathbb{R})$, the Kontsevich-Zorich cocycle $(g)_*$ is the lift of the action of g to $\mathbb{P}(\mathbf{H})$, obtained by parallel transport with respect to the Gauss-Manin connection. Furthermore, $(g)_*$ acts symplectically since it preserves the intersection form on $H^1(S, \mathbb{R})$.

For our purposes, we will in fact be concerned with the top exterior power of the Hodge bundle that is symplectic orthogonal to the tautological subbundle (spanned by $[\mathrm{Re}\,\omega]$ and $[\mathrm{Im}\,\omega]$), and we continue to call this subbundle $\mathbb{P}(\mathbf{H})$. This bundle supports an $\mathrm{SO}_2(\mathbb{R})$-invariant probability measure $\hat{\nu}$ such that, for ν-a.e ω, the conditional measure on $\mathbb{P}(\mathbf{H}_\omega)$ is the Haar measure. An Euclidean structure is in fact given by the Hodge norm (see 3.7.5 for the definition), and which we use in the sequel.

Therefore, we fix the $\mathrm{SO}_2(\mathbb{R})$-invariant Hodge norm $\|\cdot\|_{\pi(\cdot)}$ on $\mathbf{H}$. Define σ : $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{P}(\mathbf{H}) \to \mathbb{R}$ by

$$\sigma(g, \mathbf{v}) = \log \frac{\|(g)_* \mathbf{v}\|_{g\pi(\mathbf{v})}}{\|\mathbf{v}\|_{\pi(\mathbf{v})}}$$

For $\omega \in X$, let $\mathbf{v}_\omega$ in $\mathbb{P}(\mathbf{H}_\omega)$ be the projectivization of any Lagrangian subspace (of dimension $g-1$) in $\mathbb{P}(\mathbf{H})$. For $\hat{\nu}$-a.e. $\mathbf{v} = (\omega, \mathbf{v}_\omega)$, it is a consequence of the multiplicative ergodic theorem that

$$\lim_{T\to\infty} \frac{\sigma(g_T, \mathbf{v})}{T} = \sum_{i=2}^{g} \lambda_i$$

where, together with $\lambda_1 = 1$, λ_i are the top g Lyapunov exponents of the Kontsevich-Zorich cocycle, and where $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$. The top g exponents determine the entire Lyapunov spectrum by symplecticity. Note that $\lambda_2 = 0$ if and only if $\sum_{i=2}^{g} \lambda_i = 0$, since the exponents are ordered so that $\lambda_2 \geq \cdots \geq \lambda_g \geq 0$.

Our main result is the following:

Theorem 3.6.1. *There exists $\Phi_{g_\infty} \geq 0$ such that*

$$\lim_{T\to\infty} \hat{\nu}\left(\left\{\mathbf{v} \in \mathbb{P}(\mathbf{H}) \ : \ a \leq \frac{1}{\sqrt{T}}(\sigma(g_T, \mathbf{v}) - T\lambda) \leq b\right\}\right) = \frac{1}{\sqrt{2\pi\Phi_{g_\infty}}}\int_a^b \exp(-x^2/\Phi_{g_\infty})dx.$$

Remark 3.6.2. *The statement also holds for when $\Phi_{g_\infty} = 0$, in that the resulting distribution is a delta distribution.*

To prove this, we will first work with the hyperbolic Brownian motion, which is the diffusion process generated by the foliated hyperbolic Laplacian. Let ρ be a (foliated) hyperbolic Brownian motion trajectory starting at a (generic) basepoint $\omega \in X$, defined almost everywhere with respect to a probability measure $\mathbb{P}_\omega$ on the space of such trajectories W_ω. This process is in fact defined on $X^* = \mathrm{SO}_2(\mathbb{R})\backslash X$. Moreover, ρ can be lifted to $\mathrm{SL}_2(\mathbb{R})$, and is moreover defined by taking the outward radial unit tangent vector at all points. We continue to refer to the lifted path as ρ by abuse of notation. Additionally, the space X gives rise to a product space $X^W := X \otimes W$ whose fiber over each point ω in X is W_ω, and which also supports a measure $\nu_{\mathbb{P}} := \nu \otimes \mathbb{P}$, whose conditional measure over a point ω is $\mathbb{P}_\omega$. We can thus similarly define the product W-Hodge bundle $\mathbb{P}^W(\mathbf{H})$, whose fiber over each point (ω, ρ) in X^W is $\mathbf{H}_\omega$. A pair $(\rho, \mathbf{v}) \in \mathbb{P}^W(\mathbf{H})$ is thus defined to be the lift of the path ρ (starting at ω) to $\mathbb{P}^W(\mathbf{H})$, obtained by parallel transport with respect to the Gauss-Manin connection. This in turn would also give rise to a measure $\hat{\nu}_{\mathbb{P}} := \hat{\nu} \otimes \mathbb{P}$ whose conditional measure over a point $\mathbf{v}$ is $\mathbb{P}_\omega$. We therefore also have

Theorem 3.6.3. *There exists $\Phi_{\rho_\infty} \geq 0$ such that*

$$\lim_{T\to\infty} \hat{\nu}_{\mathbb{P}} \left(\left\{ (\rho, \mathbf{v}) \in \mathbb{P}^W(\mathbf{H}) \ : \ a \leq \frac{1}{\sqrt{T}}(\sigma(\rho_T, \mathbf{v}) - T\lambda) \leq b \right\} \right) \\ = \frac{1}{\sqrt{2\pi\Phi_{\rho_\infty}}} \int_a^b \exp(-x^2/\Phi_{\rho_\infty})dx.$$

Moreover, if $\lambda_2 > 0$, then $\Phi_{\rho_\infty} > 0$.

Remark 3.6.4. *Observe that for $g = 2$, our two results reduce to ones that concern the second Lyapunov exponent λ_2.*

Some ingredients of our proof include

- results of Avila-Gouëzel-Yoccoz [19] and Avila-Gouëzel [20] on the spectral gap of the leafwise hyperbolic Laplacian to show existence of a solution of the leafwise Poisson's equation (see Appendix .1),
- elementary stochastic calculus to extract and control the necessary oscillations,
- and an asymptotic estimate due to Ancona [21] to relate the geodesic flow with the Brownian motion.

3.6.1 Related results

The paper of Daniels-Deroin [38] adapts the Teichmüller dynamics methodology to more general compact Kahler manifolds, and one in which the methods in this note are applicable, provided that we can prove existence of a solution to Poisson's equation for the corresponding Laplacian. In [33], Dolgopyat-Fayad-Vinogradov prove a central limit theorem for the time integral of sufficiently regular zero-average observables of the pushforward of a small horocyclic arc by the geodesic flow, as the basepoint varies generically with respect to an ergodic P-invariant measure, where P is the upper triangular subgroup of $\mathrm{SL}_2(\mathbb{R})$ (their results are in fact much more general, cf. [33, Theorem 7.1], but we present their theorem in the $\mathrm{SL}_2(\mathbb{R})$ setting for simplicity) - it would be interesting to prove exponential mixing for the g_t-action on the Hodge bundle (more precisely for the g_t-action on a ν-strongly-irreducible $\mathrm{SL}_2(\mathbb{R})$-invariant subbundle of the Hodge bundle, and where $\hat{\nu}$ in this case would be a P-invariant measure), and then apply their result to the Kontsevich-Zorich cocycle. In [39], a mechanism to produce oscillations for the Kontsevich-Zorich cocycle is presented, where the basepoint is a fixed surface – we hope that the results presented here can be brought to bear on the scope of the result in [39], and on the limiting measures of Teichmüller horocyclic arcs as in the recent work of Chaika-Khalil-Smillie [14].

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3.7 Preliminaries

3.7.1 Translation surfaces

Let S be a Riemann surface of genus $g \geq 2$, and ω a holomorphic 1-form on S. The pair (S, ω) is said to be a translation surface, since ω gives a (degenerate) flat metric on S, and ω is invariant under translations when it is written in local coordinates. The zero set of ω characterizes the singularity set of the conical metric. The area of a translation surface is given by $\int_S \omega \wedge \overline{\omega}$. We will refer to the pair (S, ω) as just ω.

3.7.2 Moduli Space

Let $\mathcal{TH}_g$ be the Teichmüller space of unit-area translation surfaces of genus $g \geq 2$, and let $\mathcal{H}_g = \mathcal{TH}_g/\mathrm{Mod}_g$ be the corresponding moduli space, where Mod_g is the mapping class group. The space $\mathcal{H}_g$ is partitioned into strata $\mathcal{H}(\kappa) = \mathcal{H}(\kappa_1, \ldots, \kappa_n)$, which consist of unit-area translation surfaces whose singularities have cone angle $2\pi(1 + \kappa_i)$, and $\sum \kappa_i = 2g - 2$. One can also define local period coordinates on a stratum, where all changes of coordinates are given by affine maps.

3.7.3 $\mathrm{SL}_2(\mathbb{R})$ action

There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on translation surfaces and on their moduli. It is shown in [7, 8] that for any $\omega \in \mathcal{H}(\kappa)$, the closure X of $\mathrm{SL}_2(\mathbb{R}) \cdot \omega$ is an affine invariant submanifold, and supports an ergodic $\mathrm{SL}_2(\mathbb{R})$-invariant probability Lebesgue measure ν.

3.7.4 Kontsevich-Zorich cocycle

Let $\widehat{\mathbf{H}} = \mathcal{TH}_g \times H^1(S, \mathbb{R})$, and define the trivial cocycle $\widehat{(g_T)_*} : \widehat{\mathbf{H}} \to \widehat{\mathbf{H}}$ with $\widehat{(g_T)_*}(\omega, c) = (g_T\omega, c)$ for $\omega \in \mathcal{TH}_g$ and $c \in H^1(S, \mathbb{R})$. The absolute (real) Hodge bundle is given by $\mathbf{H} = \widehat{\mathbf{H}}/\text{Mod}_g$ and the Kontsevich-Zorich cocycle $(g_T)_*$ is the projection of $\widehat{(g_T)_*}$ to $\mathbf{H}$.

3.7.5 Hodge inner product and the second fundamental form

Given two holomorphic 1-forms ω_1, ω_2 in $\Omega(S)$, where $\Omega(S)$ is the vector space of holomorphic 1-forms on S, the Hodge inner product is defined to be

$$\langle \omega_1, \omega_2 \rangle := \frac{i}{2} \int_S \omega_1 \wedge \overline{\omega_2}$$

Moreover, the Hodge representation theorem implies that for any given cohomology class $c \in H^1(S, \mathbb{R})$, there is a unique holomorphic 1-form $h(c) \in \Omega(S)$, such that $c = [\text{Re}\, h(c)]$ (cf. [35]). We define the Hodge inner product for two real cohomology classes $c_1, c_2 \in H^1(S, \mathbb{R})$ as

$$A_\omega(c_1, c_2) := \langle h(c_1), h(c_2) \rangle$$

The second fundamental form B_ω is defined as

$$B_\omega(c_1, c_2) := \frac{i}{2} \int_S \frac{h(c_1)h(c_2)}{\omega^2} \omega \wedge \overline{\omega}$$

It is known that B_ω does not vanish identically in the symplectic orthogonal of the tautological subbundle on all but two orbit closures [28, 29, 31, 40]. These two orbit closures are referred to in the literature as *Eierlegende Wollmilchsau* and *Ornithorynque*, and have many special properties.

For any Lagrangian subspace $\mathbf{c}_\omega$ in $\mathbf{H}_\omega$, it also follows from the work of Forni [4] (see also [35, Corollary 2.2]) that

$$\left| \frac{d}{dt} \sigma(g_t, \mathbf{c}_\omega) \right| \leq g - 1 \tag{3.7.5.1}$$

Let $\{c_1, c_2, \ldots, c_g\}$ be a Hodge-orthonormal basis of $\mathbf{c}$ in $H^1(S, \mathbb{R})$, and let A_ω^g (resp., B_ω^g) be the corresponding representation matrix of the Hodge inner product A_ω (resp., of B_ω). The eigenvalues of B_ω^g are denoted by $\Lambda_i(\omega)$, where $1 = |\Lambda_1(\omega)| > |\Lambda_2| \geq \cdots \geq |\Lambda_g| \geq 0$. Moreover, the eigenvalues $\Lambda_i(\omega)$ are continuous, bounded functions on $\mathcal{H}_g$ (cf. [35], Lemma 2.3).

3.7.6 Foliated Hyperbolic Laplacian

The space $\mathcal{H}_g$, is foliated by the orbits of the $\mathrm{SL}_2(\mathbb{R})$-action, whose leaves are isometric to the unit cotangent bundle of the Poincaré disk $\mathbb{D}$. For $\omega \in \mathcal{H}_g$, the Teichmüller disk $L_\omega := \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}) \cdot \omega$ is isometric to $\mathbb{D}$, and so is endowed with the (foliated) hyperbolic gradient ∇_{L_ω} and hyperbolic Laplacian Δ_{L_ω}.

Remark 3.7.1. *Observe that for $\omega \in X$, the Teichmüller disk L_ω is identified with $\mathbb{D}$ via the map $(t, \theta) \mapsto \mathrm{SO}_2(\mathbb{R}) \cdot g_t r_\theta \omega$.*

Now suppose that $f : X \to \mathbb{R}$ is an $\mathrm{SO}_2(\mathbb{R})$-invariant C^∞-function in the direction of the leaf. For $\omega \in X$ and for L_ω the Teichmüller disk passing through ω, we define $\Delta f(\omega) := \Delta_{L_\omega} f|_{L_\omega}(\omega)$, where $f|_{L_\omega}$ is the restriction of f to L_ω. We also define the leafwise gradient similarly.

Observe that the Hodge inner product $A_\omega(\cdot,\cdot)$ is invariant under the action of $\mathrm{SO}_2(\mathbb{R})$, and so defines a real-analytic function on the Teichmüller disk. In the sequel, we will only work in a given Teichmüller disk, so the norm will read $(\cdot,\cdot)_z$ for a complex parameter $z \in \mathbb{D}$. For any Lagrangian $(g-1)$-plane $\mathbf{v} = (\omega, \mathbf{v}_\omega)$ in the symplectic orthogonal of the tautological subbundle (with the origin $z = 0$ corresponding to ω as in 3.7.1), define

$$\sigma(z, \mathbf{v}) := \log |\det A_z^{(g-1)}|^{1/2},$$

where $A_{z,ij}^{(g-1)} = A_z(\mathbf{v}_i, \mathbf{v}_j)$ and $\{\mathbf{v}_i\}$ is an ordered basis of $\mathbf{v}$.

Remark 3.7.2. *In fact, this is an abuse of notation since we originally lifted elements of* $\mathrm{SL}_2(\mathbb{R})$ *to the Hodge bundle. This is not an issue since the Hodge norm is* $\mathrm{SO}_2(\mathbb{R})$*-invariant.*

We recall the following fundamental fact

Theorem 3.7.3. *[4, 35] Let* $\mathbf{v}$ *be any Lagrangian subspace in the symplectic orthogonal of the tautological subbundle. We have the following equalities*

$$\Delta_{L_\omega}\sigma(z, \mathbf{v}) = 2\sum_{i=2}^{g} |\Lambda_i(z)|^2$$
$$|\nabla_{L_\omega}\sigma(z, \mathbf{v})|^2 = \left|\sum_{i=2}^{g} \Lambda_i(z)\right|^2$$

In particular, the Laplacian and the norm of the gradient are independent of the choice of a Lagrangian subspace. Moreover, we have that for ν a.e. ω,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Delta_{L_\omega} \sigma(g_t, \mathbf{v})\, dt = \int_X 2 \sum_{i=2}^{g} |\Lambda_i(\omega)|^2 d\nu = 2 \sum_{i=2}^{g} \lambda_i$$

Remark 3.7.4. *Observe that in genus $g = 2$, we have*

$$\Delta_{L_\omega} \sigma(z, \mathbf{v}_z) = 2|\nabla_{L_\omega} \sigma(z, \mathbf{v}_z)|^2 = 2|\Lambda_2(z)|^2$$

3.7.7 Harmonic measures

We say that a probability measure μ on $\mathrm{SO}_2(\mathbb{R})\backslash X$ is harmonic if for all bounded functions $f : \mathrm{SO}_2(\mathbb{R})\backslash X \to \mathbb{R}$ of class C^∞ in the leaf direction,

$$\int_{\mathrm{SO}_2(\mathbb{R})\backslash X} \Delta f(\omega)\, d\mu = \int_{\mathrm{SO}_2(\mathbb{R})\backslash X} \Delta_{L_\omega} f|_{L_\omega}(\omega)\, d\mu = 0.$$

Such a measure is also ergodic if $\mathrm{SO}_2(\mathbb{R})\backslash X$ cannot be partitioned into two union of leaves, each of which having positive μ measure. We refer the reader to the interesting paper of Lucy Garnett [41] for details and for an ergodic theorem for such measures. It is also a fact, due to Bakhtin-Martinez [42], that harmonic measures on $\mathrm{SO}_2(\mathbb{R})\backslash X$ are in one-to-one correspondence with P-invariant measures on X. This is closely related to a classical fact due to Furstenberg [43, 44] that P-invariant measures are in one-to-one correspondence with (admissible) stationary measures, and that harmonic measures are stationary. In the case of $\mathrm{SL}_2(\mathbb{R})$, these three notions are therefore closely related.

3.7.8 Hyperbolic Brownian Motion

Following the normalization used in [4] (which is a standard normalization, see also [45]), for $z = re^{i\theta}$ with $\theta \in [0, 2\pi]$, write

$$t := \frac{1}{2} \log \frac{1+r}{1-r}. \tag{3.7.8.1}$$

Since the Hodge norm is $\mathrm{SO}_2(\mathbb{R})$-invariant, it suffices to study the diffusion process generated by $\frac{1}{2}\Delta_{L_\omega}$, where the leafwise hyperbolic Laplacian in geodesic polar coordinates is given by

$$\Delta_{L_\omega} = \frac{\partial^2}{\partial t^2} + 2\coth(2t)\frac{\partial}{\partial t} + \frac{4}{\sinh^2(2t)}\frac{\partial^2}{\partial \theta^2}. \tag{3.7.8.2}$$

Moreover, let $(W_\omega^{(i)}, \mathbb{P}_\omega^{(i)})$, $i = 1, 2$, be two copies of the space of Brownian trajectories $C(\mathbb{R}^+, \mathbb{R})$ starting at the origin (with the origin corresponding to a random point ω), together with the standard Wiener measure, and such that $W_\omega^{(1)}$ and $W_\omega^{(2)}$ are independent. Set $W_\omega = W_\omega^{(1)} \times W_\omega^{(2)}$ and $\mathbb{P}_\omega = \mathbb{P}_\omega^{(1)} \times \mathbb{P}_\omega^{(2)}$. The hyperbolic Brownian motion is the diffusion process $\rho_s = (t_s, \theta_s)$ generated by the (leafwise) hyperbolic Laplacian. It follows by Ito's formula [46, Theorem VI.5.6] that the generator determines the trajectories of the diffusion process ρ_t which are solutions of the following stochastic differential equations

$$dt_s = dW_s^{(1)} + \coth(2t_s)ds \tag{3.7.8.3}$$

$$d\theta_s = \frac{2}{\sinh(2t_s)} dW_s^{(2)} \tag{3.7.8.4}$$

with $\rho_0 = 0$.

In addition, for an $\mathrm{SO}_2(\mathbb{R})$-invariant function $f : X \to \mathbb{R}$, where f is of class C^2 along $\mathrm{SL}_2(\mathbb{R})$ orbits, Ito's formula gives

$$f(\rho_T) - f(\rho_0) = \int_0^T \left(\frac{\partial}{\partial t} f(\rho_s), \frac{2}{\sinh(2t_s)} \frac{\partial}{\partial \theta} f(\rho_s) \right) \cdot \left(dW_s^{(1)}, dW_s^{(2)} \right) \tag{3.7.8.5}$$

$$+ \int_0^T \left(\frac{1}{2} \frac{\partial^2}{\partial t^2} f(\rho_s) + \frac{1}{2} 2 \coth(2t_s) \frac{\partial}{\partial t} f(\rho_s) + \frac{1}{2} \frac{4}{\sinh^2(2t_s)} \frac{\partial^2}{\partial \theta^2} f(\rho_s) \right) ds \tag{3.7.8.6}$$

$$= \int_0^T \nabla_{L_\omega} f(\rho_s) \cdot (dW_s^{(1)}, dW_s^{(2)}) + \frac{1}{2} \int_0^T \Delta_{L_\omega} f(\rho_s) ds \tag{3.7.8.7}$$

Finally, we note that the foliated heat semigroup D_t is given as follows

$$D_s f(x) := \int_X \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty f(z) p_\omega(t, s) \sinh(t) dt \, d\theta \, d\nu \tag{3.7.8.8}$$

where $p_\omega(t, s)$ is the (foliated) hyperbolic heat kernel at time s; in other words, for $x, y \in L_\omega$, this is the transition probability kernel $p_\omega(x, y; s)$, with $d_{\mathbb{D}}(x, y) = t$.

3.8 Proofs of Main Theorems

3.8.1 Distributional Convergence in Theorem 3.6.3

Recall that ρ_s is the diffusion process generated by the foliated hyperbolic Laplacian. We are interested in studying the term

$$\frac{1}{\sqrt{T}}(\sigma(\rho_T, \mathbf{v}) - T\sum_{i=2}^{g}\lambda_i). \tag{3.8.1.1}$$

Set $\lambda = \sum_{i=2}^{g}\lambda_i$. By applying Ito's formula, we obtain,

$$\frac{1}{\sqrt{T}}(\sigma(\rho_T, \mathbf{v}) - T\lambda) = \frac{\sigma(\rho_0, \mathbf{v})}{\sqrt{T}} + \frac{1}{\sqrt{T}}\int_0^T \nabla_{L_\omega}\sigma(\rho_s, \mathbf{v}) \cdot (dW_s^{(1)}, dW_s^{(2)}) \tag{3.8.1.2}$$

$$+ \frac{1}{2\sqrt{T}}\int_0^T (\Delta_{L_\omega}\sigma(\rho_s, \mathbf{v}) - 2\lambda)ds \tag{3.8.1.3}$$

It then follows then by Corollary .1.1 that the Poisson equation $\Delta_{L_\omega}u(z) = \Delta_{L_\omega}\sigma(z, \mathbf{v}) - 2\lambda$ has an L^2 solution $u(z)$ (of class C^∞ along $\mathrm{SL}_2(\mathbb{R})$ orbits), so that, by applying Ito's formula on $u(z)$, we get

$$\frac{1}{\sqrt{T}}(u(\rho_T)-u(0)) = \frac{1}{\sqrt{T}}\int_0^T \nabla_{L_\omega} u(\rho_s)\cdot(dW_s^{(1)}, dW_s^{(2)}) + \frac{1}{2\sqrt{T}}\int_0^T \Delta u(\rho_s)ds \tag{3.8.1.4}$$

$$= \frac{1}{\sqrt{T}}\int_0^T \nabla_{L_\omega} u(\rho_s)\cdot(dW_s^{(1)}, dW_s^{(2)}) \tag{3.8.1.5}$$

$$+ \frac{1}{2\sqrt{T}}\int_0^T (\Delta_{L_\omega}\sigma(\rho_s, \mathbf{v}) - 2\lambda)ds \tag{3.8.1.6}$$

So we have that

$$\frac{1}{2\sqrt{T}}\left(\int_0^T (\Delta_{L_\omega}\sigma(\rho_s, \mathbf{v}) - \lambda)ds\right) = \frac{1}{\sqrt{T}}(u(\rho_T)-u(0)) \tag{3.8.1.7}$$

$$- \frac{1}{\sqrt{T}}\int_0^T \nabla_{L_\omega} u(\rho_s)\cdot(dW_s^{(1)}, dW_s^{(2)}) \tag{3.8.1.8}$$

Define

$$M_T = \int_0^T \nabla_{L_\omega}(\sigma(\rho_s, \mathbf{v}) - u(\rho_s))\cdot(dW_s^{(1)}, dW_s^{(2)}) \tag{3.8.1.9}$$

We then have

$$\frac{1}{\sqrt{T}}(\sigma(\rho_T, \mathbf{v}) - T\sum_{i=2}^{g}\lambda_i) = \frac{1}{\sqrt{T}}(u(\rho_T) - u(0) + \sigma(\rho_0, \mathbf{v})) + \frac{1}{\sqrt{T}}M_T \tag{3.8.1.10}$$

Next, we study the quadratic variation $\langle M_T, M_T\rangle_{\hat{\nu}_{\mathbb{P}}}$. Recalling that the covariance of two Ito integrals with respect to independent Brownian motions is zero, we have:

$$\langle M_T, M_T\rangle_{\hat{\nu}_{\mathbb{P}}} = \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\left(\int_0^T \nabla_{L_\omega}(\sigma(\rho_s, \mathbf{v}) - u(\rho_s)) \cdot (dW_s^{(1)}, dW_s^{(2)})\right)^2\right] \tag{3.8.1.11}$$

$$= \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\left(\int_0^T \frac{\partial}{\partial t}(\sigma(\rho_s, \mathbf{v}) - u(\rho_s))dW_s^{(1)}\right)^2\right] \tag{3.8.1.12}$$

$$+ \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\left(\int_0^T \frac{2}{\sinh(2t_s)}\frac{\partial}{\partial \theta}(\sigma(\rho_s, \mathbf{v}) - u(\rho_s))dW_s^{(2)}\right)^2\right] \tag{3.8.1.13}$$

Applying Ito's isometry [46, Lemma VI.4.3] on the expectation of the square of the Ito integrals on the RHS yields

$$\langle M_T, M_T\rangle_{\hat{\nu}_{\mathbb{P}}} = \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^T \left(\frac{\partial}{\partial t}(\sigma(\rho_s, \mathbf{v}) - u(\rho_s))\right)^2 ds\right] \tag{3.8.1.14}$$

$$+ \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^T \left(\frac{2}{\sinh(2t_s)}\frac{\partial}{\partial \theta}(\sigma(\rho_s, \mathbf{v}) - u(\rho_s))\right)^2 ds\right] \tag{3.8.1.15}$$

$$= \mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^T |\nabla_{L_\omega}(\sigma(\rho_s, \mathbf{v}) - u(\rho_s))|^2 ds\right]. \tag{3.8.1.16}$$

Observe that $|\nabla u|^2 \in L^1(\mathrm{SO}_2(\mathbb{R})\backslash X, \nu)$ by Corollary .1.1. Therefore, by Oseledec's theorem, Fubini's theorem, and the dominated convergence theorem, we have convergence with respect to the measure $\hat{\nu}$ on $\mathbb{P}(\mathbf{H})$:

$$\Phi_{\rho_\infty} := \lim_{T\to\infty}\frac{1}{T}\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^T |\nabla_{L_\omega}(\sigma(\rho_s, \mathbf{v}) - u(\rho_s))|^2 ds\right] \tag{3.8.1.17}$$

$$= \int_{\mathbb{P}(\mathbf{H})} |B_\omega([\mathbf{E}^+(\omega)], [\mathbf{E}^+(\omega)]) - \nabla_{L_\omega}u(\omega))|^2 d\hat{\nu} \tag{3.8.1.18}$$

$$= \int_X |B_\omega([\mathbf{E}^+(\omega)], [\mathbf{E}^+(\omega)]) - \nabla_{L_\omega}u(\omega))|^2 d\nu \tag{3.8.1.19}$$

where $\mathbf{E}^+(\omega)$ denotes the top unstable Lyapunov subspace of the A-action (see also [4, Corollary 5.5]). This shows that the random cocycle converges in distribution. $\square$

3.8.2 Distributional Convergence in Theorem 3.6.1

Observe that $t_s = d_{\mathbb{D}}(0, \rho_s)$, and that it is rotationally invariant. We will need the following useful lemma:

Lemma 3.8.1. *[46, Lemma VII.7.2.1] For all $\omega \in X$, there exists an $\mathbb{P}_\omega$-almost everywhere converging process η_s such that $t_s = W_s^{(1)} + s + \eta_s$.*

Proof. It is a classical fact that $t_s \to \infty$ $\mathbb{P}_\omega$-almost everywhere. This implies that $\lim_{s\to\infty} \coth(2t_s) = 1$ almost everywhere. Setting $\eta_s := t_s - W_s^{(1)} - s$, so that, together with 3.7.8.3, we get

$$\eta_s = \int_0^s (\coth(2t_s) - 1)ds = \int_0^s \frac{2ds}{e^{4t_s} - 1},$$

which converges almost everywhere, as desired. $\square$

Next, it will be crucial to stop the radial process before it exits the region bounded by a circle of geodesic radius T, and so for each T, we define the stopping time τ_T as follows

$$\tau_T := \inf\{s > 0 \ : \ T = d_{\mathbb{D}}(0, \rho_s)\} \tag{3.8.2.1}$$

$$= \inf\{s > 0 \ : \ T = W_s^{(1)} + s + \eta_s\} \tag{3.8.2.2}$$

where the second equality follows by Lemma 3.8.1. Next, we will need the following lemma:

Lemma 3.8.2. *For all $\omega \in X$, we have $\lim_{T\to\infty} \tau_T/T = 1$ $\mathbb{P}_\omega$-almost everywhere. Moreover, we have that as $T \to \infty$, $\tau_t \to \infty$ $\mathbb{P}_\omega$-almost everywhere.*

Proof. Observe that we have $\tau_T = T - W^{(1)}_{\tau_T} - \eta_{\tau_T}$. The lemma then follows immediately from the definition of the stopping time and the law of the iterated logarithm. □

See also [47, Lemma 4.2] for related and interesting results on this stopping time.

Recall that $\mathbb{P}_\omega$ is the Wiener measure on the space of all Brownian trajectories W_ω starting at the origin (corresponding to the random point ω). Let $\mathbb{P}^\theta_\omega$ be the Wiener measure on the space W^θ_ω corresponding to all paths starting at the origin and conditioned to exit at the point $e^{i\theta}$ in $\partial\mathbb{D}^2$. To relate the conditioned process ρ^θ_s to the unconditioned process ρ_s, we will need the following lemma:

Lemma 3.8.3.

$$\mathbb{P}_\omega = \frac{1}{2\pi}\int_0^{2\pi} \mathbb{P}^\theta_\omega \, d\theta \tag{3.8.2.3}$$

Proof. Recall that W_ω is the space of all hyperbolic Brownian motion trajectories starting at the origin, with $\mathbb{P}_\omega$ the corresponding Wiener measure. There exists a map $\Theta : W_\omega \to \partial\mathbb{D}^2$, defined $\mathbb{P}_\omega$-almost everywhere, such that $\Theta(\rho) = \rho_\infty$, where ρ_∞ is the limit point of ρ on $\partial\mathbb{D}^2$. It is a classical fact that the pushforward measure $\Theta_*(\mathbb{P}_\omega)$ equals Leb, where Leb is the normalized Lebesgue measure on $[0, 2\pi]$. We also recall that the foliated process is in fact defined on $\mathrm{SO}_2(\mathbb{R})\backslash X$ and that $\hat{\nu}$ is $\mathrm{SO}_2(\mathbb{R})$-invariant, and so our disintegration claim follows. $\square$

Remark 3.8.4. *See also [48, Lemma 8] for a short potential theoretic proof (using Doob's h-process) of this fact. The approach to proving the CLT in [48], with the aid of a stopping time, is what we will essentially follow in the sequel, though in our case the proof here is simpler, in view of the Lipschitz property of the Kontsevich-Zorich cocycle and Ancona's estimate.*

Remark 3.8.5. *It is worth repeating and adapting what is written in the introduction in view of the application of the conditioned process in the sequel. The conditioned process is in fact defined on $X^* = \mathrm{SO}_2(\mathbb{R})\backslash X$. Moreover, ρ^θ can be lifted to $\mathrm{SL}_2(\mathbb{R})$, and is moreover defined by taking the outward radial unit tangent vector at all points. We continue to refer to the lifted path as ρ^θ by abuse of notation. Additionally, the space X gives rise to a product space $X^{W^\theta} := X \otimes W^\theta$ whose fiber over each point ω in X is W^θ_ω, and which also supports a measure $\nu_{\mathbb{P}^\theta} := \nu \otimes \mathbb{P}^\theta$, whose conditional measure over a point ω is $\mathbb{P}^\theta_\omega$. We can thus similarly define the product W^θ-Hodge bundle $\mathbb{P}^{W^\theta}(\mathbf{H})$, whose fiber over each point (ω, ρ^θ) in X^{W^θ} is $\boldsymbol{H}_\omega$. A pair $(\rho^\theta, \mathbf{v}) \in \mathbb{P}^{W^\theta}(\boldsymbol{H})$ is thus defined to be the lift of the path ρ^θ (starting at ω) to $\mathbb{P}^{W^\theta}(\boldsymbol{H})$, obtained by parallel transport with respect to the Gauss-Manin connection. This in turn would also give rise to a measure $\hat{\nu}_{\mathbb{P}^\theta} := \hat{\nu} \otimes \mathbb{P}^\theta$ whose conditional measure over a point $\mathbf{v}$ is $\mathbb{P}^\theta_\omega$.*

We recall the following fundamental result due to Ancona [21] (see also [49, Lemma 4.1]):

Theorem 3.8.6. *[21, Théorème 7.3] For all $\omega \in X$, and $\mathbb{P}_\omega$-almost all paths ρ starting at ω, we have that $d_{\mathbb{D}}(\rho_0\rho_\infty, \rho_T) = O(\log T)$ as $T \to \infty$, where $\rho_0\rho_\infty$ is the geodesic ray* *with* $\rho_0 \in \mathbb{D}$ *and* $\rho_\infty \in \partial\mathbb{D}$.

Now observe that our aim is to study

$$\Sigma^g(T,[a,b]) := \hat{\nu}\left(\left\{\mathbf{v} \in \mathbb{P}(\mathbf{H}) \; : \; a \le \frac{1}{\sqrt{T}}(\sigma(g_T,\mathbf{v}) - T\lambda) \le b\right\}\right) \tag{3.8.2.4}$$

as $T \to \infty$.

Let

$$\Sigma^\rho(T,[a,b]) := \hat{\nu}_{\mathbb{P}}\left(\left\{(\rho,\mathbf{v}) \in \mathbb{P}^W(\mathbf{H}) \; : \; a \le \frac{1}{\sqrt{T}}(\sigma(\rho_{\tau_T},\mathbf{v}) - T\lambda) \le b\right\}\right) \tag{3.8.2.5}$$

Lemma 3.8.7. *The quantity*

$$|\Sigma^g(T,[a,b]) - \Sigma^\rho(T,[a,b])| \to 0 \tag{3.8.2.6}$$

as $T \to \infty$, $\mathbb{P}_\omega$*-almost everywhere and for all* $\omega \in X$.

Proof. By applying the disintegration in Lemma 3.8.3, 3.8.2.5 is also equal to

$$\Sigma^\rho(T,[a,b]) = \text{Leb} \otimes \hat{\nu}_{\mathbb{P}^\theta}\left(\left\{(\theta,\rho^\theta,\mathbf{v}) \in [0,2\pi] \otimes \mathbb{P}^{W^\theta}(\mathbf{H}) \; : \right.\right. \tag{3.8.2.7}$$

$$\left.\left. a \le \frac{1}{\sqrt{T}}(\sigma(\rho^\theta_{\tau_T},\mathbf{v}) - T\lambda) \le b\right\}\right) \tag{3.8.2.8}$$

$$= \text{Leb} \otimes \hat{\nu}_{\mathbb{P}^\theta}\left(\left\{(\theta,\rho^\theta,\mathbf{v}) \in [0,2\pi] \otimes \mathbb{P}^{W^\theta}(\mathbf{H}) \; : \right.\right. \tag{3.8.2.9}$$

$$\left.\left. a \le \frac{1}{\sqrt{T}}(\sigma(g_T r_\theta,\mathbf{v}) - T\lambda) + \frac{1}{\sqrt{T}}(\sigma(\rho^\theta_{\tau_T},\mathbf{v}) - \sigma(g_T r_\theta,\mathbf{v})) \le b\right\}\right) \tag{3.8.2.10}$$

Theorem 3.8.6 applied on τ_T gives that, for all $\omega \in X$, $d_{\mathbb{D}}(g_T r_\theta \cdot 0, \rho^\theta_{\tau_T}) = O(\log \tau_T)$ $\mathbb{P}^\theta_\omega$-almost everywhere as $T \to \infty$. Together with Lemma 3.8.2, the lemma now follows by the Lipschitz property of the Kontsevich-Zorich cocycle (by the derivative bound in 3.7.5.1). $\square$

Therefore, it suffices to study the limiting distribution of the quantity

$$\frac{1}{\sqrt{T}}(\sigma(\rho_{\tau_T}, \mathbf{v}) - T\lambda).$$

Observe that we have that for all $\omega \in X$, and $\mathbb{P}_\omega$-almost everywhere, $\tau_t \to \infty$ as $T \to \infty$. By applying the stopping time identity $T = \tau_T + W^{(1)}_{\tau_T} + \eta_{\tau_T}$, a straightforward calculation shows the following equality:

$$\begin{aligned}
\frac{1}{\sqrt{T}}(\sigma(\rho_{\tau_T}, \mathbf{v}) - T\lambda) = &- \frac{1}{\sqrt{T}}\eta_{\tau_T}\lambda && (3.8.2.11)\\
&- \frac{1}{\sqrt{T}}W^{(1)}_{\tau_T}\lambda && (3.8.2.12)\\
&+ \frac{1}{\sqrt{T}}(\sigma(\rho_{\tau_T}, \mathbf{v}) - \tau_T\lambda) && (3.8.2.13)
\end{aligned}$$

So this reduces the proof of the theorem to controlling three terms on the RHS of the previous equality. First, we can observe that 3.8.2.11 clearly converges to zero $\mathbb{P}_\omega$-almost everywhere by Lemma 3.8.1. Next, it follows by Lemma 3.8.2 that

$$\lim_{T\to\infty} \frac{\lambda}{\sqrt{T}} W^{(1)}_{\tau_T} \xrightarrow{d} W^{(1)}_{\lambda^2}$$

and in particular the asymptotic variance of 3.8.2.12 is λ^2. The asymptotic variance of

3.8.2.13 converges to Φ_{ρ_∞} by Theorem 3.6.3 and Lemma 3.8.2, together with the simple observation that

$$\frac{1}{\sqrt{T}}(\sigma(\rho_{\tau_T}, \mathbf{v}) - \tau_T \lambda) = \frac{\sqrt{\tau_T}}{\sqrt{T}} \frac{1}{\sqrt{\tau_T}}(\sigma(\rho_{\tau_T}, \mathbf{v}) - \tau_T \lambda).$$

The following lemma concerns the covariance of the terms 3.8.2.12 and 3.8.2.13, and shows that it converges almost everywhere:

Lemma 3.8.8. $Cov_{\hat{\nu}_{\mathbb{P}}}\left(\frac{1}{\sqrt{T}} M_{\tau_T}, -\frac{\lambda}{\sqrt{T}} W^{(1)}_{\tau_T}\right) \to -\lambda^2$

Proof. We will first need the following fact, which follows by [4, Lemma 3.1], together with the observation that $\Delta_{L_\omega}(\sigma - u) = 2\lambda$, we have,

$$\frac{\partial}{\partial t} \frac{1}{2\pi} \int_0^{2\pi} (\sigma(z, \mathbf{v}) - u(z)) d\theta = \frac{1}{\sinh(2t)} \int_0^t \frac{1}{2\pi} \int_0^{2\pi} \Delta_{L_\omega}(\sigma - u) d\theta \sinh(2r) dr \tag{3.8.2.14}$$

$$= \lambda \frac{\cosh(2t) - 1}{\sinh(2t)} \tag{3.8.2.15}$$

$$= \lambda \tanh(t) \tag{3.8.2.16}$$

We are now ready to calculate the covariance. We have

$$\mathrm{Cov}_{\hat{\nu}_{\mathbb{P}}}\left(\frac{1}{\sqrt{T}}M_{\tau_T}, -\frac{\lambda}{\sqrt{T}}W^{(1)}_{\tau_T}\right) \tag{3.8.2.17}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda}{T}\int_0^{\tau_T}\nabla_{L_\omega}(\sigma(\rho_s,\mathbf{v})-u(\rho_s))\cdot(dW_s^{(1)},dW_s^{(2)})\int_0^{\tau_T}dW_s^{(1)}\right] \tag{3.8.2.18}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda}{T}\int_0^{\tau_T}\frac{\partial}{\partial t}(\sigma(\rho_s,\mathbf{v})-u(\rho_s))dW_s^{(1)}\int_0^{\tau_T}dW_s^{(1)}\right] \tag{3.8.2.19}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda}{T}\int_0^{\tau_T}\frac{\partial}{\partial t}(\sigma(\rho_s,\mathbf{v})-u(\rho_s))ds\right] \tag{3.8.2.20}$$

$$= -\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\frac{\lambda}{T}\int_0^{T}\frac{\partial}{\partial t}(\sigma(\rho_s,\mathbf{v})-u(\rho_s))ds\right]+o(1) \tag{3.8.2.21}$$

$$= -\frac{\lambda^2}{T}\mathbb{E}_{\hat{\nu}_{\mathbb{P}}}\left[\int_0^{T}\tanh(t_s)ds\right]+o(1) \tag{3.8.2.22}$$

$$\to -\lambda^2 \tag{3.8.2.23}$$

where 3.8.2.19 follows by the independence of $W_s^{(1)}$ and $W_s^{(2)}$, and where 3.8.2.20 follows by an application of Ito's inner product (a more general case of Ito's isometry, which follows by applying the polarization identity), which also holds for our stopping time – in fact, Ito's isometry holds for stochastic integrals with infinite time horizon, and so it also follows for our defined stopping time (see also [46, Lemma VI.4.3]). We also note that 3.8.2.21 holds thanks to Lemma 3.8.2 and the identity within its proof. Finally, 3.8.2.22 holds thanks to 3.8.2.16, together with the rotational invariance of the hyperbolic heat kernel. □

To conclude the proof of Theorem 3.6.1, we observe that since the Brownian motion has normally distributed independent increments, a linear combination of Brownian motion terms is also normally distributed. This, together with convergence of the asymptotic covariance in Lemma 3.8.8, completes the proof, and in particular we have that the asymptotic variance Φ_{g_∞} is

$$\Phi_{g_\infty} = \Phi_{\rho_\infty} + \lambda^2 + 2 \lim_{T\to\infty} \operatorname{Cov}\left(\frac{2}{\sqrt{T}} M_{\tau_T}, -\frac{\lambda\sqrt{2}}{\sqrt{T}} W^{(1)}_{\tau_T} \right) \tag{3.8.2.24}$$

$$= \Phi_{\rho_\infty} + \lambda^2 - 2\lambda^2 = \Phi_{\rho_\infty} - \lambda^2 \tag{3.8.2.25}$$

□

3.9 Positivity of the variance

3.9.1 Random cocycle

Recall that 3.8.2.25 says that $\Phi_{g_\infty} = \Phi_{\rho_\infty} - \lambda^2$, and so we also have the following important corollary:

Corollary 3.9.1. *If $\lambda_2 > 0$, then $\Phi_{\rho_\infty} > 0$*

Proof. Since, by construction, $\Phi_{g_\infty} \geq 0$, and we have that $\Phi_{\rho_\infty} \geq \lambda^2 > 0$, and it is clear that, since $\lambda = \sum \lambda_i$, we have $\lambda^2 \geq \lambda_2^2$. □

3.9.2 Deterministic cocycle

Remark 3.9.2. *It is not clear to us how our formulas can be leveraged to deduce positivity of the variance for the deterministic cocycle. Our claim therefore is that the deterministic cocycle converges in distribution with a $\sqrt{T}$ normalization, and we hope that this result could be useful to specifically address the question of positivity of the variance via different* *methods.*

.1 Solving Poisson's equation

The purpose of this section is to prove a straightforward corollary of the spectral gap of the foliated Laplacian due to Avila-Gouëzel, Avila-Gouëzel-Yoccoz, which will be key to the proofs of our main theorems.

Corollary .1.1. *[19, 20] For $f \in L^2(\mathrm{SO}_2(\mathbb{R})\backslash X, \nu)$ with $\int_X f d\mu = 0$, we can find $u \in L^2(\mathrm{SO}_2(\mathbb{R})\backslash X, \nu)$ with $\Delta u = f$. As a consequence, $|\nabla u|^2$ is also in $L^2(\mathrm{SO}_2(\mathbb{R})\backslash X, \nu)$.*

We follow closely the notation in Avila-Gouëzel, [20, Section 3.4], and we refer to their paper for more details and references. In particular, following their notation, and for ξ varying in the space Ξ of all unitary irreducible representations of $\mathrm{SL}_2(\mathbb{R})$, let H_ξ be a family of representations. For us, we will be concerned with the following spherical decomposition

$$L^2(\mathrm{SO}_2(\mathbb{R})\backslash X, \nu) \simeq \int_\Xi H_\xi^{\mathrm{SO}_2(\mathbb{R})} dm(\xi) \tag{.1.0.1}$$

where $H_\xi^{\mathrm{SO}_2(\mathbb{R})}$ is the set of $\mathrm{SO}_2(\mathbb{R})$-invariant vectors contained in H_ξ, and m a measure on Ξ. It is a fact that the spectrum of the Laplacian Δ on $L^2(\mathrm{SO}_2(\mathbb{R})\backslash X, \nu)$ is equal to the set $\{(1 - s(\xi)^2)/4\}$, where ξ is a spherical representation in supp m. Finally, taking into account the direct integral in .1.0.1, we also have that the L^2 norm of a function

$f : X \to \mathbb{R}$ is given as

$$\|f\|^2 = \int \|f_\xi\|^2_{H_\xi} \, dm(\xi). \tag{.1.0.2}$$

Proof of Corollary .1.1. Let

$$u := \int_\Xi \frac{4}{(1 - s(\xi)^2)} f_\xi dm(\xi) \tag{.1.0.3}$$

Since the spherical unitary irreducible representations are the principal (for which s is purely imaginary) and the complementary ones, we also have

$$\begin{aligned}
\|u\|^2 &= \int_\Xi \left\| \frac{4}{(1 - s(\xi)^2)} f_\xi \right\|^2_{H_\xi} dm(\xi) && (.1.0.4)\\
&= \int_{\Xi_{\text{comp}}} \left\| \frac{4}{(1 - s(\xi)^2)} f_\xi \right\|^2_{H_\xi} dm(\xi) + \int_{\Xi_{\text{princ}} \subset [0,\infty]} \left\| \frac{4}{(1 + y^2)} f_y \right\|^2_{H_y} dy && (.1.0.5)
\end{aligned}$$

We also have that the spectra $\sigma(\Delta) \cap (0, 1/4) = \sigma(\Omega) \cap (0, 1/4)$, and that the spectral measures coincide, which follows since, in the interval $(0, 1/4)$, the complementary series representations are all spherical. In particular, it follows from the work of Avila-Gouëzel and Avila-Gouëzel-Yoccoz that there are finitely many representations in the complementary series (where $s(\xi)$ lies in $(0, 1)$) that appear in the decomposition of $L^2(\mathrm{SO}_2(\mathbb{R})\backslash X, \nu)$ into irreducible representations of $\mathrm{SL}_2(\mathbb{R})$, and they have finite multiplicity. This immediately gives

$$\int_{\Xi_{\text{comp}}} \left\| \frac{4}{(1 - s(\xi)^2)} f_\xi \right\|^2_{H_\xi} dm(\xi) < \infty \tag{.1.0.6}$$

Moreover, and recalling again that for irreducible unitary representations of the principal series, $s(\xi)$ is equal to iy. As a consequence, we also have

$$\int_{\Xi_{\text{princ}} \subset [0,\infty]} \left\| \frac{4}{(1+y^2)} f_y \right\|_{H_y}^2 dy \leq 16 \int_{\Xi_{\text{princ}} \subset [0,\infty]} \|f_y\|_{H_y}^2 \, dy < \infty \tag{.1.0.7}$$

giving us that the solution u is in L^2.

To show that $|\nabla u|^2$ is also in L^2, we recall that for X, Y the generators of the geodesic flow (X) and the orthogonal geodesic flow (Y), respectively, we have that $\Delta u = -(X^2 + Y^2)u$ since u is an $\mathrm{SO}_2(\mathbb{R})$-invariant function. We claim that $\langle \Delta u, u \rangle_{L^2} = \|\nabla u\|_{L^2}^2$. Since X and Y are volume-preserving, and therefore also skew-adjoint, we have that

$$\langle \Delta u, u \rangle_{L^2} = \langle Xu, Xu \rangle_{L^2} + \langle Yu, Yu \rangle_{L^2} \tag{.1.0.8}$$

$$= \|\nabla u\|_{L^2}^2 \tag{.1.0.9}$$

as desired. □

Remark .1.2. *In fact, more can be said of the radial and angular derivatives of u via the representation theory of $\mathrm{SL}_2(\mathbb{R})$, and we refer to Flaminio-Forni [50] for details in that direction.*

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